Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-09-08

Conceptual questions

Question 1. See the back of this handout.

Computations

First, some questions on Chapter 12.

Problem 1. Are there any vectors $\mathbf{v}$ such that $\langle 1,2,1\rangle \times \mathbf{v} = \langle 3,1,-5\rangle$? If so, find all of them. If not, explain why not.

Then do the same question, but for $\langle 1,2,1\rangle \times \mathbf{v} = \langle 3,1,5\rangle$.

Problem 2. Let L_1 be the line passing through $A(1,-2,4)$ and $B(2,1,3)$, and let L_2 be the line passing through $C(0,3,-3)$ and $D(2,4,1)$.

Are L_1, L_2 parallel, skew, or intersecting? If they intersect, where do they intersect? If not, how far apart are they?

The next ones introduce some §13.2 content into the mix.

Problem 3. Find the point where the curves $\mathbf{r}_1(t) = \langle 2t, 2-2t, 3+t^2\rangle$ and $\mathbf{r}_2(t) = \langle 6-2t, 2t-4, t^2\rangle$ intersect. Then compute the angle formed by the two curves at their point of intersection.

Problem 4. Let $\mathbf{r}_1(t)$ be as in the preceding problem, and let H be the plane $x+y-2z+6=0$. The curve given by $\mathbf{r}_1$ intersects H twice. Find these two points, and determine the angle at which the curve meets the plane at each point.

The following are solutions to the problem

"Find the distance d between the point $P(1,-2,2)$ and the line $\mathbf{r}(t) = \langle 3+3t, 2-t, 5t\rangle$."

Figure out what is happening in each one.

(a) Solution 1:

$$\begin{aligned} D^2 &= (2+3t)^2 + (4-t)^2 + (5t-2)^2 \\ &= 35t^2 - 16t + 24 \\ \frac{\mathrm{d}}{\mathrm{d}t}(D^2) &= 70t - 16 = 0 \\ t &= 8/35 \\ d = D_{\min} &= \sqrt{35(8/35)^2 - 16(8/35) + 24} = 2\sqrt{194/35}. \end{aligned}$$

(b) Solution 2:

$$\begin{aligned} 3(x-1) - (y+2) + 5(z-2) &= 0 \\ 3x - y + 5z - 15 &= 0 \\ 3(3+3t) - (2-t) + 5(5t) - 15 &= 0 \\ t &= 8/35 \\ d &= \sqrt{(3+3(8/35)-1)^2 + (2-(8/35)+2)^2 + (5(8/35)-2)^2} = 2\sqrt{194/35}. \end{aligned}$$

(c) Solution 3:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ \langle -2,-4,2\rangle \times \langle 3,-1,5\rangle &= \langle -18,16,14\rangle \\ |\langle -18,16,14\rangle| &= 2\sqrt{194} \\ |\langle 3,-1,5\rangle| &= \sqrt{35} \\ d &= 2\sqrt{194/35}. \end{aligned}$$

(d) Solution 4:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ (\langle 3,-1,5\rangle \times \langle -2,-4,2\rangle) \times \langle 3,-1,5\rangle &= \langle 94,132,-30\rangle = 2\langle 47,66,-15\rangle \\ \frac{\langle 47,66,-15\rangle \cdot \langle -2,-4,2\rangle}{|\langle 47,66,-15\rangle|} &= -388/\sqrt{6790} \\ d &= |-388/\sqrt{6790}| = 2\sqrt{194/35}. \end{aligned}$$

(e) Solution 5:

$$\begin{aligned} \langle 2+3t, 4-t, 5t-2\rangle \cdot \langle 3,-1,5\rangle &= 0 \\ 35t - 8 &= 0 \\ t &= 8/35 \\ d &= \sqrt{(3+3(8/35)-1)^2 + (2-(8/35)+2)^2 + (5(8/35)-2)^2} = 2\sqrt{194/35} \end{aligned}$$

(f) Solution 6:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ \frac{\langle 3,-1,5\rangle \cdot \langle -2,-4,2\rangle}{\langle 3,-1,5\rangle \cdot \langle 3,-1,5\rangle}\langle 3,-1,5\rangle &= \frac{8}{35}\langle 3,-1,5\rangle \\ \langle -2,-4,2\rangle - \frac{8}{35}\langle 3,-1,5\rangle &= \langle -\frac{94}{35}, -\frac{132}{35}, \frac{6}{7}\rangle \\ |\langle -\frac{94}{35}, -\frac{132}{35}, \frac{6}{7}\rangle| &= 2\sqrt{194/35}. \end{aligned}$$

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. We went over all of the solutions in class.

Answers to computations

Problem 1. Let $\mathbf{v} = \langle x, y, z \rangle$. We get the system of equations

$$\begin{aligned} 2z - y &= 3 \\ x - z &= 1 \\ y - 2x &= -5 \end{aligned}$$

It turns out this has infinitely many solutions; we can take x as a free variable for instance. The solutions together form a line:

$$x = t, \;\; y = 2t - 5, \;\; z = t - 1.$$

On the other hand, if we do this with $\langle 3, 1, 5 \rangle$ instead, we would find no solutions. One way to see this without solving a system of equations is by noting that the cross product *must* be orthogonal to both inputs, whereas $\langle 3, 1, 5 \rangle$ is not orthogonal to $\langle 1, 2, 1 \rangle$ (their dot product is nonzero).

Problem 2. Here are the equations of the two lines:

$$\begin{aligned} \mathbf{L}_1(t) &= \langle 1, -2, 4 \rangle + t\langle 1, 3, -1 \rangle, \\ \mathbf{L}_2(s) &= \langle 0, 3, -3 \rangle + s\langle 2, 1, 4 \rangle. \end{aligned}$$

Inspection of their slope vectors shows that they are not parallel, so they are either skew or intersecting.

By attempting to equate their coordinates and trying to solve for s, t, you will find no solutions, meaning that the lines do not intersect. Hence they must be skew. (Alternatively, once you compute the distance between the lines and find that it is nonzero, that also eliminates the case they are intersecting.)

The distance between them is then given by

$$\left| \operatorname{comp}_{\langle 1,3,-1 \rangle \times \langle 2,1,4 \rangle} (\langle 1, -2, 4 \rangle - \langle 0, 3, -3 \rangle) \right| = \left| \operatorname{comp}_{\langle 13,-6,-5 \rangle} \langle 1, -5, 7 \rangle \right| = \boxed{8/\sqrt{230}}.$$

Problem 3. They intersect at $\mathbf{r}_1(1) = \mathbf{r}_2(2) = \langle 2, 0, 4 \rangle$. Hence we need to evaluate the angle between

$$\mathbf{r}_1'(1) = \langle 2, -2, 2 \rangle, \mathbf{r}_2'(2) = \langle -2, 2, 4 \rangle$$

which works out to $\pi/2$ since their dot product is 0. (In other words, the curves meet perpendicularly.)

Problem 4. To find the points of intersection, we simply substitute:

$$(2t) + (2 - 2t) - 2(3 + t^2) + 6 = 0$$

hence $t = -1, 1$, so the points of intersection are $(-2, 4, 4)$ and $(2, 0, 4)$.

For the point $(-2, 4, 4)$ corresponding to $t = -1$, we have that $\mathbf{r}_1'(-1) = \langle 2, -2, -2 \rangle$ whereas a normal vector for the plane is $\mathbf{n} = \langle 1, 1, -2 \rangle$. The angle formed between these vectors is $\cos^{-1}(\sqrt{2}/3)$, but the angle we want is $\boxed{\pi/2 - \cos^{-1}(\sqrt{2}/3)}$.

For the other point corresponding to $t = 1$, we have $\mathbf{r}_1'(1) = \langle 2, -2, 2 \rangle$ which forms an angle of $\cos^{-1}(-\sqrt{2}/3)$ with $\mathbf{n}$. So the angle we want is $\boxed{\cos^{-1}(-\sqrt{2}/3) - \pi/2}$ (which is actually the same as the other angle).